The n-Wiener Polynomials of the Cartesian Product of a Complete Graph with some Special Graphs

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ABSTRACT

The n-Wiener polynomials of the Cartesian products of a complete graph $K_t$ with another complete graph $K_r$, a star graph $S_r$, a complete bipartite graph $K_{r,s}$, a wheel $W_r$, and a path graph $P_r$ are obtained in this paper. The n-diameters and the n-Wiener indices of $K_t \times K_r$, $K_t \times S_r$, $K_t \times K_{r,s}$, $K_t \times W_r$ and $K_t \times P_r$ are also obtained.

Keywords: n-distance, n-diameter, n-index, n-Wiener polynomial.

1. Introduction.

We follow the terminology of [5] and [6]. Let $v$ be a vertex of a connected graph $G$ and let $S$ be an (n-1)-subset of vertices of $V(G)$, $n \geq 2$, then the n-distance $d_n(v,S)$ is defined as follows[7]

$$d_n(v,S) = \min \{d(v,u) : u \in S\}.$$ ...(1.1)

Sometimes, we refer to the n-distance of the pair $(v,S)$ in $G$ by $d_n(v,S \mid G)$. The n-diameter $\text{diam}_n G$ of $G$ is defined by

$$\text{diam}_n G = \max \{d_n(v,S) : v \in V(G), S \subseteq V(G), |S| = n-1\}.$$ ...(1.2)

It is clear that for all $2 \leq m \leq p$,

$$\text{diam}_m G \leq \text{diam}_n G \leq \text{diam}_G.$$ ...(1.3)

The n-Wiener index of $G$ denoted by $W_n(G)$ is defined as

$$W_n(G) = \sum_{(v,S)} d_n(v,S),$$ ...(1.4)
where the summation is taken over all pairs \((v,S)\) for which \(v \in V(G)\), \(S \subseteq V(G)\) and \(|S|=n-1\). The **n-average distance** \(\mu_n(G)\) is defined as

\[
\mu_n(G) = \frac{W_n(G)}{p\left(\binom{p-1}{n-1}\right)}, \quad 3 \leq n \leq p.
\]  

(1.5)

Let \(v\) be any vertex of \(G\), then the **n-distance of \(v\)** denoted \(d_n(v; G)\) or simply \(d_n(v)\) is defined as

\[
d_n(v) = \sum_{S \subseteq V(G)} d_n(v, S), \quad |S| = n-1.
\]  

(1.6)

The Wiener polynomial of \(G\) with respect to the \(n\)-distance, which is called **\(n\)-Wiener polynomial** and defined as below.

**Definition 1.1.[2]** Let \(C_n(G,k)\) be the number of pairs \((v,S), |S|=n-1, 3 \leq n \leq p\), such that \(d_n(v,S)=k\), for each \(0 \leq k \leq \delta_n\). Then, the **\(n\)-Wiener polynomial** \(W_n(G;x)\) is defined by

\[
W_n(G;x) = \delta_n \sum_{k=0}^{\delta_n} C_n(G,k)x^k,
\]  

(1.7)

in which \(\delta_n\) is the \(n\)-diameter of \(G\).

One may easily see [2] that for \(3 \leq n \leq p\), the number of all \((v,S)\) pairs is

\[
p\left(\binom{p-1}{n-1}\right), \quad \text{and [1]}
\]

\[
\sum_{k=1}^{\delta_n} C_n(G,k) = p\left(\binom{p-1}{n-1}\right), \quad C_n(G,0) = p\left(\binom{p-1}{n-2}\right),
\]  

(1.8)

\[
C_n(G,1) = p\left(\binom{p-1}{n-1}\right) - \sum_{v \in V(G)} \left(\binom{p-1}{n-1}\deg_G(v)\right)
\]  

(1.9)

**Definition 1.2[1]** Let \(v\) be a vertex of \(G\), and let \(C_n(v,G,k)\) be the number of \((n-1)\)-subsets of vertices of \(G\) such that \(d_n(v,S|G)=k\), for \(n \geq 3\), \(0 \leq k \leq \delta_n\). Then, the **\(n\)-Wiener polynomial of vertex \(v\)**, denoted by \(W_n(v,G;x)\) is defined as

\[
W_n(v,G;x) = \sum_{k=0}^{\infty} C_n(v,G,k)x^k.
\]  

(1.10)

It is clear that for all \(k \geq 0\),

\[
\sum_{v \in V(G)} C_n(v,G,k) = C_n(G,k),
\]  

(1.11)

and

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\[ \sum_{v \in V(G)} W_n(v,G,x) = W_n(G;x). \quad \ldots(1.12) \]

There are many classes of graphs \( G \) in which for each \( k, 1 \leq k \leq \delta_n \), \( C_n(v,G,k) \) is the same for every vertex \( v \in V(G) \); such graphs are called \([1] \) vertex-n-distance regular. If \( G \) is of order \( p \) and it is vertex-n-distance regular, then
\[ W_n(G;x) = pW_n(v,G;x), \quad \ldots(1.13) \]
where \( v \) is any vertex of \( G \).

The authors of references \([2],[3]\) and \([4]\) obtained the \( n \)-Wiener polynomials of some special graphs and some types of composite graphs. In this paper, we obtain \( n \)-Wiener polynomials of the Cartesian products \( K_t \times K_r \), \( K_t \times S_r \), \( K_t \times K_{r,s} \), \( K_t \times W_r \) and \( K_t \times P_r \).

2. The Cartesian Product of a Complete Graph and a Star

Let \( K_t \) be a complete graph with \( V(K_t) = \{u_1,u_2,\ldots,u_t\} \), and \( S_r \) be a star of center \( v_0 \) and end vertices \( v_1,v_2,\ldots,v_{r-1} \). Each vertex of \( K_t \times S_r \) is an ordered pair \( (u_i,v) \), \( 1 \leq i \leq t, 0 \leq j \leq r-1 \). Let \( K_t^j \) be the clique graph \([6]\) of order \( t \) of vertex set \( \{ (u_i,v_j) : i=1,2,\ldots,t, 0 \leq j \leq r-1 \} \). The graph \( K_t \times S_r \) is depicted in Fig. 2.1.

![Fig. 2.1. The graph \( K_t \times S_r \).](image)

It is clear that \( 0 \leq d((u_i,v_j),(u_i,v_m)) \leq 3 \). Thus,
\[ \text{diam}_n K_t \times S_r \leq \text{diam} K_t \times S_r \leq 3. \]

**Proposition 2.1.** For \( t \geq 2, r \geq 3 \), the \( n \)-diameter of \( K_t \times S_r \) is given by
\[ \text{diam}_n K_t \times S_r = \begin{cases} 3, & \text{if } 2 \leq n \leq (t-1)(r-2)+1, \\ 2, & \text{if } 1+(t-1)(r-2)+1 < n \leq t(r-1), \\ 1, & \text{if } t(r-1) < n \leq rt. \end{cases} \]
Proof. The proof is clear from Fig. 2.1. ■

The following theorem gives us the n-Wiener polynomial of $K_t \times S_r$. It is clear that the order of $K_t \times S_r$ is $p=rt$.

**Theorem 2.2.** For $t \geq 2$, $r \geq 3$, $3 \leq n \leq rt$,

$$W_n(K_t \times S_r;x) = p \left( \begin{array}{c} p-1 \\ n-2 \end{array} \right)x + \left( p \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) - t \left( \begin{array}{c} p-t-1 \\ n-1 \end{array} \right) \right)x^2 + t(r-1)x^3.$$ 

Proof. It is clear that each vertex of $K_t^0$ is of degree $r+t-2$, and each vertex of $K_t^j$, $1 \leq j \leq r-1$, is of degree $t$. Therefore, by (1.9) we obtained $C_n(K_t \times S_r,1)$ as given in the theorem.

To find $C_n(K_t \times S_r,3)$, we notice that there are $(t-1)(r-2)$ vertices each of distance 3 from each vertex $(u,v)$ of $K_t^j$, $1 \leq j \leq r-1$. Thus,

$$C_n(K_t \times S_r,3) = t(r-1) \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right).$$

Finally, by (1.8) and Proposition 2.1, we get

$$C_n(K_t \times S_r,2) = p \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) - C_n(K_t \times S_r,1) - C_n(K_t \times S_r,3)$$

$$= t \left( \begin{array}{c} p-t-r+1 \\ n-1 \end{array} \right) + (r-1) \left( \begin{array}{c} p-t-1 \\ n-1 \end{array} \right) - (r-1) \left( \begin{array}{c} p-2t+r+2 \\ n-1 \end{array} \right).$$

Hence, the proof. ■

**Corollary 2.3.** For $t \geq 2$, $r \geq 3$, $3 \leq n \leq rt$,

$$W_n(K_t \times S_r) = p \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} p-t-1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} p-t-r+1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} p-2t+r+2 \\ n-1 \end{array} \right).$$

3. The Cartesian Product of Complete Graphs

Let $K_t$ and $K_r$ be disjoint complete graphs, and let $(u_1,v_1),(u_2,v_2) \in V(K_t \times K_r)$, then it is clear that

$$diam K_t \times K_r = 2.$$ 

Thus,

$$diam_n K_t \times K_r \leq 2, \quad 2 \leq n \leq rt.$$
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If \( u_1 \neq u_2 \) and \( v_1 \neq v_2 \), then \((u_1, v_1), (u_2, v_2)\) are non-adjacent in \( K_t \times K_r \); and \((u_1, v_1), (u_1, v_2)\), \((u_2, v_2)\) is a path of length 2. Therefore,
\[
d((u_1, v_1), (u_2, v_2)) = 2.
\]
The degree of each vertex \((u_1, v_1)\) is \( r + t - 2 \). Thus, the number of vertices of distance 2 from \((u_1, v_1)\) is \( rt - r - t + 1 \). Hence, we have the following result.

**Proposition 3.1.** For \( t, r \geq 2 \),
\[
d((u_1, v_1), (u_2, v_2)) = \begin{cases} 2 & \text{if } 2 \leq n \leq rt - r - t + 2, \\ 1 & \text{if } rt - t + 3 \leq n \leq rt. \end{cases}
\]

Now, we find the n-Wiener polynomial of \( K_t \times K_r \).

**Theorem 3.2.** For \( r, t \geq 2 \), \( 3 \leq n \leq rt \)
\[
W_n(K_t \times K_r; x) = rt \begin{pmatrix} n-1 \choose 2 \end{pmatrix} + rt \begin{pmatrix} n-1 \choose n-1 \end{pmatrix} x + rt \begin{pmatrix} n-1 \choose n-1 \end{pmatrix} x^2.
\]

**Proof.** It is clear that \( K_t \times K_r \) is vertex-n-distance regular. Thus,
\[
C_n(K_t \times K_r, 2) = rt C_n((u_1, v_1), K_t \times K_r, 2).
\]
Since the number of vertices of distance 2 from \((u_1, v_1)\) is \( rt - t + 1 \), and there is no vertex of distance more than 2 from \((u_1, v_1)\), then
\[
C_n((u_1, v_1), K_t \times K_r, 2) = \begin{pmatrix} n-1 \choose n-1 \end{pmatrix}.
\]
The constant term and the coefficient of \( x \) follow from (1.8) and (1.9).■

**Corollary 3.3.** For \( t, r \geq 2 \), \( 3 \leq n \leq rt \),
\[
W_n(K_t \times K_r) = rt \begin{pmatrix} n-1 \choose n-1 \end{pmatrix} + rt \begin{pmatrix} n-1 \choose n-1 \end{pmatrix} x.\]

4. The Cartesian Product of a Complete Graph and a Complete Bipartite Graphs

Let \( K_{r,s} \) be a complete bipartite graph of bipartite sets of vertices
\( V_1 = \{v_1, v_2, \ldots, v_r\} \), \( V_2 = \{w_1, w_2, \ldots, w_s\}; r \geq s \), and let \( V(K_t) = \{u_1, u_2, \ldots, u_t\} \),
then it is clear that in \( K_t \times K_{r,s} \)
\[
d((u_i, v_h), (u_j, v_k)) = 3 \text{ when } i \neq j, h \neq k,
\]
because there is a shortest path
\((u_i, v_h), (u_j, v_h), (u_j, w), (u_i, v_k), w \in V_2.\)
Similarly,
\[
d((u_i, w_h), (u_j, w_k)) = 3 \text{ when } i \neq j, h \neq k.\]
Moreover,
\[
d((u_i, v_h), (u_i, v_k)) = d((u_i, w_h), (u_i, w_k)) = 2.
\]
Therefore,
\[ \text{diam } K_t \times K_{r,s} = 3, \]
and so
\[ \text{diam}_n K_t \times K_{r,s} \leq 3, \quad 2 \leq n \leq p, \quad p = t(r+s). \]
For any vertex \((u_i, v_h)\), the number of vertices of distance 3 from \((u_i, v_h)\) in \(K_t \times K_{r,s}\) is \((t-1)(r-1)\). Similarly, there are \((t-1)(s-1)\) vertices of distance 3 from \((u_i, w_k)\). Moreover, the degree of each vertex of \(K_t \times K_{r,s}\) is either \(r+t-1\) or \(s+t-1\).

Thus, we have the following result.

**Proposition 4.1.** For \(t, r, s \geq 2\), \(r \geq s\), then the \(n\)-diameter of \(K_t \times K_{r,s}\) is given
\[
\begin{align*}
\text{diam}_n K_t \times K_{r,s} & = \begin{cases} 
3, & \text{for } 2 \leq n \leq t-r+2, \\
2, & \text{for } tr-r+3 \leq n \leq p-t+s, \\
1, & \text{for } p-t+s+1 \leq n \leq p.
\end{cases}
\end{align*}
\]

The next theorem determines the \(n\)-Wiener polynomial of \(K_t \times K_{r,s}\).

**Theorem 4.2.** For \(t, r, s \geq 2\), \(3 \leq n \leq p\), \(p = t(r+s)\),
\[
W_n(K_t \times K_{r,s}; x) = p \left( \begin{array}{c} p-1 \\ n-2 \end{array} \right) + \sum_{k=1}^{p} \left[ \begin{array}{c} p-1 \\ n-1 \end{array} \right] - rt \left[ \begin{array}{c} p-t-1 \\ n-1 \end{array} \right] - st \left[ \begin{array}{c} p-t-1 \\ n-1 \end{array} \right] \right] x^2
\]
\[
+ \left[ rt \left( \begin{array}{c} r-t+s+1 \\ n-1 \end{array} \right) - st \left( \begin{array}{c} a-t+s+1 \\ n-1 \end{array} \right) \right] x^3.
\]

**Proof.** \(C_n(K_t \times K_{r,s}, 0)\) and \(C_n(K_t \times K_{r,s}, 1)\) are obtained from (1.8) and (1.9). To find the other coefficients, we notice that \(C_n((a,b), K_t \times K_{r,s}, k)\) is the same for every vertex \((a,b) \in V(K_t) \times V_1\), and \(C_n((c,d), K_t \times K_{r,s}, k)\) is the same for every vertex \((c,d) \in V(K_t) \times V_2\), for \(k = 2, 3\). Since the number of vertices of distance 3 from vertex \((a,b)\) is \((t-1)(r-1)\), and the number of vertices of distance 3 from vertex \((c,d)\) is \((t-1)(s-1)\), then we get the coefficient of \(x^3\) as given in the statement of the theorem.
Finally, \(C_n(K_t \times K_{r,s}, 2)\) is obtained using the relation (1.8) and the coefficients already obtained. This completes the proof.

**Corollary 4.3.** For \(t, r, s \geq 2\), and \(3 \leq n \leq p\) in which \(p = t(r+s)\),
\[
W_n(K_t \times K_{r,s}) = p \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + rt \left( \begin{array}{c} r-t \+ st \left( \begin{array}{c} a-t \+ rt \left( \begin{array}{c} r-t+s+1 \\ n-1 \end{array} \right) - st \left( \begin{array}{c} a-t+s+1 \\ n-1 \end{array} \right) \right] \right] x^2
\]
\[
+ \left[ rt \left( \begin{array}{c} r-t+s+1 \\ n-1 \end{array} \right) - st \left( \begin{array}{c} a-t+s+1 \\ n-1 \end{array} \right) \right] x^3.
\]
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\[ +st\left[ \binom{p-r-t}{n-1} + \binom{n-t-r+1}{n-1} \right]. \]

**Proof.** The n-Wiener index is obtained from \( W_n(K_t \times K_{r,s}; x) \) by taking the derivative with respect to \( x \), and then put \( x=1 \), and simplified the expression. ■

5. The Cartesian Product of a Complete Graph and a Wheel

Let \( W_r \) be a wheel of order \( r \geq 4 \) and let its center be denoted by \( v_0 \) and its other vertices be \( v_1, v_2, \ldots, v_{r-1} \). Moreover, let \( V(K_t) = \{ u_1, u_2, \ldots, u_t \} \). The order of \( K_t \times W_r \) is \( p = rt \), and in \( K_t \times W_r \)

\[ \text{deg}(u_i, v_j) = t+2, \text{ for } 1 \leq i \leq t, 1 \leq j \leq r-1, \]

\[ \text{deg}(u_i, v_0) = t+r-2. \]

One can easily see that in \( K_t \times W_r \)

\[ d((u_i, v_0), (u_j, v_h)) = 2, \text{ for } i \neq j, h \neq 0, \]

\[ d((u_i, v_h), (u_j, v_m)) = 3, \text{ for } i \neq j, h \neq m, h, m \neq 0, \]

because \( (u_i, v_h), (u_j, v_h), (u_j, v_0), (u_j, v_m) \) is a shortest \( (u_i, v_h) - (u_j, v_m) \) when \( v_h v_m \notin W_r \). Thus,

\[ \text{diam } K_t \times W_r = 3, \text{ when } r \geq 5. \]

Thus, for \( r \geq 5, t \geq 2, \)

\[ \text{diam } K_t \times W_r \leq 3. \]

Since for each vertex \( (u_i, v_h) \), \( 1 \leq i \leq t, h \neq 0 \) there are \( (t-1)(r-4) \) vertices of distance 3 from \( (u_i, v_h) \), and \( \text{deg}(u_i, v_h) = t+2 \), then we have the following result.

**Proposition 5.1.** For \( t \geq 2, r \geq 5 \), the n-diameter of \( K_t \times W_r \) is given by

\[ \text{diam}_n K_t \times W_r = \begin{cases} 3, & \text{for } 2 \leq n \leq 1 + (t-1)(r-4), \\ 2, & \text{for } 2 + (t-1)(r-4) \leq n \leq p-t-2, \\ 1, & \text{for } p-t-1 \leq n \leq p. \end{cases} \]

The following theorem gives us the n-Wiener polynomial of \( K_t \times W_r \).

**Theorem 5.2.** For \( t \geq 2, r \geq 5, 3 \leq n \leq p, p = tr \)

\[ W_n(K_t \times W_r; x) = p \left( \binom{p-1}{n-2} + \binom{p-1}{n-1} - t(r-1) \binom{p-r-3}{n-1} - t \binom{p-r-1}{n-1} \right) x \]

\[ + \left[ t(r-1) \left( \binom{p-r-3}{n-1} + t \binom{p-r-1}{n-1} - t(r-1) \binom{p-r-4}{n-1} \right) \right] x^2 \]

\[ + t(r-1) \binom{p-r-4}{n-1} x^3. \]

**Proof.** The coefficients of \( x^0 \) and \( x \) are obtained using (1.8) and (1.9). To obtain the coefficient of \( x^3 \), we notice that for any \( (u_i, v_0) \), \( 1 \leq i \leq t \) and every
(n-1)-set of vertices $S$, $d_n((u_i,v_0),S) \leq 2$. But for every vertex $(u_i,v_j)$, $1 \leq i \leq t$, $1 \leq j \leq r-1$, there are $(t-1)(r-4)$ vertices each of distance 3 from $(u_i,v_j)$. Therefore, there are \( \binom{p-r-4t+4}{n-1} \) sets $S$, $|S|=n-1$, such that $d_n((u_i,v_j),S)=3$. Thus,

\[
C_n(K_t \times W_r, 3) = t(r-1) \binom{p-r-4t+4}{n-1}.
\]

We obtain $C_n(K_t \times W_r, 2)$ by using (1.8). Hence, the proof. ■

**Corollary 3.4.3.** For $t \geq 2$, $r \geq 5$ and $3 \leq n \leq rt$,

\[
W_n(K_t \times W_r) = p \binom{p-1}{n-2} + t(r-1) \binom{p-r-3}{n-1} + t \binom{p-r-1}{n-1} + (r-1) \binom{p-r-4t+4}{n-1}.
\]

**Proof.** The proof follows from Theorem 5.2 and the fact $W_n(K_t \times W_r) = W_n(K_t \times W_r, 1)$. ■

**6. The Cartesian Product of a Path and a Complete Graph**

Let $P_r$, $r \geq 2$ be a path graph of order $r$ and $P_r: v_1,v_2,\ldots,v_r$, and let $V(K_t) = \{u_1, u_2, \ldots, u_t\}$, $t \geq 3$.

The Cartesian product $K_t \times P_r$ is shown in Fig. 6.1. The following proposition determines the n-diameter of $K_t \times P_r$.

**Proposition 6.1.** For $t \geq 2$, $r \geq 3$, $2 \leq n \leq rt$,

\[
\text{diam}_n K_t \times P_r = r + 1 - \left\lceil \frac{n}{t} \right\rceil.
\]

![Diagram of the Cartesian product of a path and a complete graph](image)
**Proof.** From Fig. 6.1, we notice that \(d_n((u,v),S), |S|=n-1\) has maximum value when \((u,v)\) is one of the vertices in \(A_1 \cup A_r\), where
\[
A_i=\{(u_j,v_i): j=1,2,\ldots,t\},
\]
and \(S\) is the \((n-1)\)-set of vertices farthest from \((u,v)\) in \(K_t \times P_r\). Thus, we may take the vertex \((u_1,v_r)\), and \(S\) consisting of vertices of \(A_1,A_2,\ldots,A_i\) and some vertices of \(A_{i+1}-\{(u_1,v_{i+1})\}\) when
\[
it \leq n-1 \leq t(i+1)-1;
\]
and when
\[
2 \leq n \leq t, \text{ then } S \subseteq A_1-\{(u_1,v_1)\}.
\]
In the last case,
\[
diam_n K_t \times P_r = r;
\]
and in general case of \(n\),
\[
diam_n K_t \times P_r = r-i, \text{ } it+1 \leq n \leq (i+1)t.
\]
One can easily see that
\[
i=\left\lceil \frac{n}{t} \right\rceil-1.
\]
Hence, in any case of the value of \(n\),
\[
diam_n K_t \times P_r = r+1-\left\lceil \frac{n}{t} \right\rceil. \blacksquare
\]

Now, we obtain the \(n\)-Wiener polynomial of \(K_t \times P_r\) in the following two theorems.

**Theorem 6.2.** Let \(r=2s\), \(s \geq 1\), \(t \geq 3\) and \(3 \leq n \leq rt\). Then
\[
W_n(K_t \times P_r;x)=\sum_{k=0}^{\delta} C_n(K_t \times P_r,k)x^k,
\]
where
\[
C_n(K_t \times P_r,0)=rt \begin{pmatrix} r-1 \\ n-2 \end{pmatrix},
\]
\[
C_n(K_t \times P_r,1)=rt \begin{pmatrix} r-1 \\ n-1 \end{pmatrix} -2t \begin{pmatrix} r-t-1 \\ n-1 \end{pmatrix} -t(r-2) \begin{pmatrix} r-t-2 \\ n-1 \end{pmatrix},
\]
for \(2 \leq k \leq s \)
\[
C_n(K_t \times P_r,k)=2t[\sum_{i=1}^{k-1} \binom{a+i-k}{n-1} -\binom{a-k}{n-1} ] +2 \begin{pmatrix} a+2t-k-1 \\ n-1 \end{pmatrix} -\begin{pmatrix} a-k \\ n-1 \end{pmatrix}
\]
\[
+(s-k)\left\{ \begin{pmatrix} a+2r-k-1 \\ n-1 \end{pmatrix} -\begin{pmatrix} a-k \\ n-1 \end{pmatrix} \right\},
\]
for \(k \geq s+1 \)
\[ C_n(K_t \times P_r, k) = 2t \sum_{i=1}^{s} \binom{a+t}{n-1} - \binom{a}{n-1}, \]

in which \( \alpha = p - t(k-1) - 1 \).

**Proof.** \( C_n(K_t \times P_r, 0) \) and \( C_n(K_t \times P_r, 1) \) are obtained from (1.8) and (1.9). For \( 2 \leq k \leq \delta_n \) we shall consider three cases for the values of \( k \).

(1) If \( 2 \leq k < s \), then for each \( 1 \leq i \leq s \) the number of vertices of distance \( k \) from any vertex, say \((u_j, v_i)\), of \( A_i \) is \( t \) and the number of vertices of distance more than \( k \) from \((u_j, v_i)\) is \( p - t(i+k-1) - 1 = \alpha - ti \) when \( 1 \leq i \leq k-1 \) which gives us

\[ a = \sum_{j=1}^{k-1} \sum_{i=1}^{n-1} \binom{a-ti}{n-1-j} \] \hspace{1cm} ...(6.1)

If \( i = k \), then there are \( 2t-1 \) vertices of distance \( k \) from \((u_j, v_i)\), and there are \( p - t(2k-1) - 2 \) vertices of distance more than \( k \). This gives us

\[ b = \sum_{j=1}^{n-1} \binom{p-2k+t-1}{n-1-j} \] \hspace{1cm} ...(6.2)

If \( k+1 \leq i \leq s \), then there are \( 2t \) vertices of distance \( k \) from \((u_j, v_i)\) and there are \( p - t(2k-1) - 2 \) vertices of distance more than \( k \). This gives us

\[ c = \sum_{i=k+1}^{s} \sum_{j=1}^{n-1} \binom{p-2k+t-2}{n-1-j} \] \hspace{1cm} ...(6.3)

Since \( r = 2s \) and each \( A_i \) consists of \( t \) vertices,

\[ C_n(K_t \times P_r, k) = 2t(a+b+c) \] when \( 2 \leq k < s \).

(2) If \( k = s \), then using the same reasoning as in case (1) we find that (6.1) and (6.2) are true for this case, and (6.3) does not hold. Thus,

\[ C_n(K_t \times P_r, k) = 2t(a+b) \] when \( k = s \).

(3) If \( k \geq s+1 \), then it is clear that both (6.2) and (6.3) do not hold. Thus,
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\[ C_n(K_t \times P_r, k) = 2\alpha \text{ when } k \geq s+1. \]

Substituting \( a, b \) and \( c \), we get the required results. ■

**Theorem 6.3.** Let \( r = 2s+1, s \geq 1, t \geq 3 \) and \( 3 \leq n \leq rt \).

Then

\[ W_n(K_t \times P_r; x) = \sum_{k=0}^{\delta_n} C_n(K_t \times P_r, k)x^k, \]

where

\[ C_n(K_t \times P_r, 0) = rt \binom{rt}{n-2}, \]

\[ C_n(K_t \times P_r, 1) = rt \binom{rt-1}{n-1} - 2t \binom{rt-1}{n-1} - t(r-2) \binom{rt-2}{n-1}, \]

for \( 2 \leq k \leq s \),

\[ C_n(K_t \times P_r, k) = 4t \sum_{i=1}^{k-1} \left\{ \binom{\alpha+rt-i}{n-1} - \binom{\alpha-i}{n-1} + \binom{\alpha+2rt-i-1}{n-1} - \binom{\alpha-i}{n-1} \right\} \]

\[ + t(r-2k) \left\{ -\binom{\alpha+2rt-i-1}{n-1} + \binom{\alpha-i}{n-1} \right\}, \]

for \( k = s+1, \)

\[ C_n(K_t \times P_r, k) = 2t \sum_{i=1}^{s} \left\{ \binom{\alpha+rt-i}{n-1} - \binom{\alpha-i}{n-1} \right\} + t \binom{2r-2}{n-1}, \]

for \( s+1 < k < \delta_n, \)

\[ C_n(K_t \times P_r, k) = 2t \sum_{i=1}^{s} \left\{ \binom{\alpha+rt-i}{n-1} - \binom{\alpha-i}{n-1} \right\}, \]

in which

\[ \alpha = p - t(k-1) - 1. \]

**Proof.** The proof of \( C_n(K_t \times P_r, k) \) for \( k \neq s+1 \) is similar to that for even \( r \) given in Theorem 6.2. For \( k = s+1 \) we add the number of pairs \( (u_j, v_{s+1}), S \) of \( n \)-distance \( s+1 \), which equals \( \binom{2r-2}{n-1} \) for each \( 1 \leq j \leq t. \) ■
REFERENCES


